Example: Find the Riemann sum formula for $\int_{-1}^{2} e^{x^2}(2x^2+1)\,dx$.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x, \text{ where } \Delta x = \frac{b-a}{n} = \frac{3}{n}$$

& $x_k = a + k\Delta x = -1 + k\left(\frac{3}{n}\right) = \left(-1 + \frac{3}{n}k\right)$

$$\int_{-1}^{2} e^{x^2}(2x^2+1)\,dx = \boxed{\lim_{n\to\infty} \sum_{k=1}^{n} e^{\left(-1+\frac{3k}{n}\right)^2}\left(2\cdot\left(-1+\frac{3k}{n}\right)^2+1\right)\frac{3}{n}}.$$

Examples: ① find $\int_0^1 \frac{1}{x^2+1}\,dx$

② find $\int_0^{\pi/8} \sec(2x)\tan(2x)\,dx$

① Use FTC: $(\arctan(x))' = \frac{1}{x^2+1}$

$$\Rightarrow \int_0^1 \frac{1}{x^2+1}\,dx = \arctan(x)\Big|_0^1 = \arctan(1) - \arctan(0)$$

$$= \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}.$$

② $\int_0^{\pi/8} \sec(2x)\tan(2x)\,dx$

Thinking: $(\sec(x))' = \sec(x)\tan(x)$.

$(\sec(2x))' = \sec(2x)\tan(2x)\cdot 2$

$\Rightarrow \left(\left(\frac{1}{2}\sec(2x)\right)' = \sec(2x)\tan(2x).\right)$

$\frac{1}{2}A = \frac{A}{2}$

$\frac{1}{2}\cdot\frac{A}{1} = \frac{A}{2}$

$\Rightarrow \int_0^{\pi/8} \sec(2x)\tan(2x)\,dx$

$= \left(\frac{1}{2}\sec(2x)\right)\Big|_0^{\pi/8}$

$= \frac{1}{2}\sec\left(2\cdot\frac{\pi}{8}\right) - \frac{1}{2}\sec(2\cdot 0)$

$= \frac{1}{2}\frac{1}{\underbrace{\cos\left(\frac{\pi}{4}\right)}_{\frac{\sqrt{2}}{2}}} - \frac{1}{2\underbrace{\cos(0)}_{1}} = \frac{1}{\sqrt{2}} - \frac{1}{2}$

$= \frac{\sqrt{2}}{2} - \frac{1}{2} = \boxed{\frac{\sqrt{2}-1}{2}}$

Tricks of Guessing Antiderivatives.

- Substitution: Chain Rule, backwards.

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

$$\therefore \int f'(\underbrace{g(x)}) \underbrace{g'(x)\,dx} = f(g(x)) + C$$

$$\boxed{\begin{aligned} &\text{let } u = g(x) \\ &\quad du = g'(x)\,dx \end{aligned}}$$

$$\int f'(u)\,du = f(u) + C = f(g(x)) + C$$

eg. $d(\sin\theta) = \cos\theta \; d\theta$

$d(2^p) = (2^p \cdot \ln(2))\,dp$

Examples ① $\int 2x\cos(x^2)\,dx$

Subs: $u = \begin{pmatrix}\text{inside}\\ \text{fcn}\end{pmatrix} = x^2$

$du = 2x\,dx$

$$\rightarrow = \int \cos(u)\,du = \sin(u) + C$$

$$= \boxed{\sin(x^2) + C}$$

Check: $\left[\sin(x^2)\right]' = \cos(x^2) \cdot 2x$ ✓

② $\displaystyle\int_{x=0}^{x=\sqrt{\pi}/2} 2x\cos(x^2)\,dx = ?$

let $u = x^2$

$du = 2x\,dx$

Bounds

$u = x^2 = \left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{4}$

$u = x^2 = (0)^2 = 0$

$$= \int_{u=0}^{u=\pi/4} \cos(u)\,du$$

$$= \sin(u)\Big|_0^{\pi/4} = \sin\left(\frac{\pi}{4}\right) - \sin(0)$$

$$= \frac{1}{\sqrt{2}} - 0 = \boxed{\frac{1}{\sqrt{2}}}.$$

3. $\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}\,dx = ?$

$u = \sqrt{x}+2$

$du = \frac{1}{2}x^{-1/2}\,dx = \frac{1}{2\sqrt{x}}\,dx$

$2\,du = 2\left(\frac{1}{2\sqrt{x}}\,dx\right) = \frac{1}{\sqrt{x}}\,dx$

$= \int u^3\, 2\,du = \int 2u^3\,du$

$\left(\frac{1}{2}u^4\right)' = 2u^3$

$\left(2\cdot\frac{u^4}{4}\right)$

$\int u^n\,du$

$\frac{u^{n+1}}{n+1} + C$ if $n \neq -1$

$\int \frac{1}{u}\,du = \ln|u| + C$

$= \frac{1}{2}u^4 + C$

$= \boxed{\frac{1}{2}(\sqrt{x}+2)^4 + C}$

Check: $\left[\frac{1}{2}(\sqrt{x}+2)^4+C\right]'$

$$= \frac{1}{\cancel{2}}\cdot \cancel{4}(\sqrt{x}+2)^3\cdot\frac{1}{\cancel{2}\sqrt{x}} = \frac{(\sqrt{x}+2)^3}{\sqrt{x}}. \checkmark$$

Method 2 $\displaystyle\int \frac{(\sqrt{x}+2)^3}{\sqrt{x}}\,dx$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$\int \frac{(\sqrt{x})^3+3(\sqrt{x})^2\cdot 2+3(\sqrt{x})\cdot 2^2+2^3}{\sqrt{x}}\,dx$$

$$= \int \frac{x^{3/2}+6x+12x^{1/2}+8}{x^{1/2}}\,dx$$

$$= \int \left(x+6x^{1/2}+12+8x^{-1/2}\right)dx$$

$$= \frac{x^2}{2}+6\cdot\frac{x^{3/2}}{3/2}+12x+\frac{8x^{1/2}}{1/2}+C$$

$$\overset{6\cdot\frac{2}{3}}{}$$

$$= \boxed{\frac{x^2}{2}+4x^{3/2}+12x+16x^{1/2}+C}$$

$$\int_{x=0}^{1} \frac{x}{2+3x^2}\,dx$$

let $u = 2+3x^2$

$du = 6x\,dx$

$\frac{1}{6}du = \frac{1}{6}6x\,dx = x\,dx$

If $\int_0^1 \frac{x}{(2+3x)^2}dx$

Then let

$u = 2+3x$

$du = 3dx$

Bounds

$u = 2+3x^2 = 5$

$u = 2+3x^2 = 2$

$$= \int_{u=2}^{5} \frac{\frac{1}{6}du}{u}$$

$$= \frac{1}{6}\int_{u=2}^{5} \frac{du}{u} = \frac{1}{6}\ln|u| \Big|_2^5$$

$$= \boxed{\frac{1}{6}\ln(5) - \frac{1}{6}\ln(2)}$$